J. Fluid Mech. (2003), *vol.* 495, *pp.* 135–143.
DOI: 10.1017/S0022112003006281 Printed in the United Kingdom

Pressure–velocity correlations and scaling exponents in turbulence

By VICTOR YAKHOT

Department of Aerospace and Mechanical Engineering, Boston University, Boston, MA 02215, USA

(Received 2 July 2003 and in revised form 15 August 2003)

It is shown that each structure function $S_{n,m}(r)$, of order $n+m$, in strong turbulence is characterized by its own dissipation scale $\eta_{n,m}$. In the limit $n \to \infty$, the dissipation scale $\eta_{n,0} = O(Re^{-1})$, which is much smaller than the Kolmogorov scale $\eta = O(Re^{-3/4})$, Re being the large-scale Reynolds number. This result has implications for the resolution requirements of direct simulations of turbulence. A new rigorous dynamic constraint relating scaling exponents of the structure functions to the codimension of the most singular features of turbulence is derived. A modification of the model by Gotoh & Nakano (2003) for the pressure–velocity correlations, based on the Bernoulli equation, is proposed. This proposal leads to an analytic expression for the scaling exponents of velocity structure functions.

1. Introduction

One of the central topics of turbulence theory is the behaviour of velocity structure functions $S_{n,m}$, of order $n+m$, defined as

$$S_{n,m}(r) \equiv \langle U^n V^m \rangle \equiv \langle (\delta u)^n (\delta v)^m \rangle, \tag{1}$$

where

$$\delta u_i = u_i(\boldsymbol{x}+\boldsymbol{r}) - u_i(\boldsymbol{x}).$$

Here, the displacement vector $\boldsymbol{r}$ is parallel to the x-axis and u and v are the x and y components of the velocity vector $\boldsymbol{u}$, respectively. In the inertial range where viscosity $\nu \to 0$ and the displacement r is much smaller than the integral scale L $(r/L \to 0)$ and much larger than the dissipation scale η $(r/\eta \to \infty)$, the structure functions are assumed to follow the algebraic relations

$$S_{n,m} = A_{n,m}\left(\frac{r}{L}\right)^{\xi_{n,m}}. \tag{2}$$

The amplitudes $A_{n,m}$ include the mean dissipation rate $\epsilon = \nu\overline{(\partial_i u_j)(\partial_i u_j)}$ and the integral scale L. (In what follows, to simplify notation, we will set $\epsilon = L = 1$.)

To define the dissipation scales $\eta_{n,m}$ of various structure functions, we note that, in the limit $r/\eta_{n,m} \to 0$, the functions $S_{n,m}$ are of the form $\overline{(\partial u/\partial x)^n(\partial v/\partial x)^m} r^{m+n}$. At the dissipation scale,

$$S_{n,m}(\eta_{n,m}) \approx \overline{\left(\frac{\partial u}{\partial x}\right)^n \left(\frac{\partial v}{\partial x}\right)^m} \eta_{n,m}^{m+n} = A_{m,n}\eta_{n,m}^{\xi_{n,m}}. \tag{3}$$

Choosing a Reynolds-number-independent constant $\mathscr{C} \gg 1$ enables us to specify the inertial range: $L \gg r \to \mathscr{C}\eta_{n,m} \to 0$ when $Re \to \infty$. The possibility that the constant $\mathscr{C}$

can depend upon the moment order (n, m) will be considered below in some detail. In this range Kolmogorov's (1941) theory of three-dimensional turbulence produced two exact relations for the third-order moments $S_{3,0}$ and $S_{1,2}$:

$$S_{3,0} = -\tfrac{4}{5}\epsilon r; \qquad S_{3,0}/S_{1,2} = 3.$$

In addition, kinematic considerations (Kolmogorov 1941) give

$$\frac{\partial S_{2,0}}{\partial r} + \frac{2S_{2,0}}{r} = 2\frac{S_{0,2}}{r}.$$

Using this result in conjunction with (2), we obtain

$$\frac{\xi_{2,0}+2}{2} = \frac{A_{0,2}}{A_{2,0}}.$$

Kolmogorov took a further step: based on his result for the third-order longitudinal structure function $S_{3,0}(r)$, he concluded that $U = O((\epsilon r)^{1/3})$ and proposed a general law: $S_{n,0} \propto r^{n/3}$ (Kolmogorov 1941; Monin & Yaglom 1971; Frisch 1995). However, recent experimental work testing Kolmogorov's theory has shown that the scaling exponents $\xi_{n,m}$ deviate from the Kolmogorov scaling relations $\xi_{n,0} = n/3$ and cannot be derived on dimensional grounds. (For the most recent compilation of the magnitudes of the scaling exponents obtained in both physical and numerical experiments, see Kurien & Sreenivasan 2001). This aspect of three-dimensional turbulence, now called anomalous scaling, has remained for some time a major challenge to turbulence theory.

Below, based on the exact equation for the moments of velocity difference (Yakhot 1998, 2001; see also Hill 2001, who rederived them using a different method), we present in § 2 some new rigorous results for the scaling exponents $\xi_{2n,0}$. It will be shown that the problem of anomalous scaling in turbulence can be reduced to the problem of conditional expectation values of the pressure gradient difference for the fixed values of velocity increment. A model proposed in § 3 for this conditional expectation leads to a closed expression for magnitudes of the scaling exponents $\xi_{n,0}$, in good agreement with experimental data. The paper concludes with a summary of results in § 4.

2. Equation for structure functions and some consequences

The equations for the generating function $Z = \langle \exp(\lambda \cdot \delta \boldsymbol{u})\rangle$ and, as a consequence, for all moments of velocity differences $S_{n,m} = (\partial^{n+m}/\partial\lambda_x^n\lambda_y^m)\, Z(\lambda = 0)$ in d-dimensional turbulence were derived in Yakhot (1998, 2001). For the even-order inertial-range structure functions $S_{2n,0}$, this equation gives

$$\frac{\partial S_{2n,0}}{\partial r} + \frac{d-1}{r}S_{2n,0} = \frac{(d-1)(2n-1)}{r}S_{2n-2,2} - (2n-1)\langle \delta p_x (\delta u)^{2n-2}\rangle, \qquad (4)$$

where $p_x \equiv \partial_x p$. Equation (4) was investigated experimentally in Kurien & Sreenivasan (2001) and numerically in Gotoh & Nakano (2003). Note that this equation does not contain the dissipation contribution $D = \nu\overline{(\nabla^2_{x+r}u(x+r) - \nabla^2_x u(x))U^{2n-2}} \rightarrow 0$ in the limit $\nu \rightarrow 0$. This is clear on the grounds of symmetry of the Navier–Stokes equations which are invariant under the transformation $u \rightarrow -u$ and $x \rightarrow -x$. However, the equation does involve the pressure contributions

$$\langle \delta p_x (\delta u)^{2n-2}\rangle = \langle [p_{x'}(\boldsymbol{x}+\boldsymbol{r}) - p_x(\boldsymbol{x})](\delta u)^{2n-2}\rangle, \qquad (5)$$

where ($x' = x + r$). These terms are very hard to close from first principles. Below we will need the pressure difference correlation function defined as

$$S_{p2} = \overline{(\delta p)^2} = A_p r^{\xi_{\mathscr{P}}}. \tag{6}$$

The accepted value for $\xi_{\mathscr{P}}$ is roughly 4/3 (Monin & Yaglom 1971), but the precise value will not be important for many of our present considerations. It is easy to see that $C_p = \overline{(\partial_x p(x))^2} = \lim_{r\to\eta_p} \partial_r^2 S_{p2}(r) \approx \eta_p^{\xi_{\mathscr{P}}-2}$. As $Re \to \infty$, C_p diverges for all values of $\xi_{\mathscr{P}} < 2$, in particular as $\eta^{-4/3}$ if $\xi_{\mathscr{P}} = 4/3$.

In the inertial range where $r/\eta \geqslant \mathscr{C} \gg 1$, we have

$$\overline{\left(\frac{\partial p(x)}{\partial x} - \frac{\partial p(x')}{\partial x'}\right)^2} = 2\overline{\left(\frac{\partial p(x)}{\partial x}\right)^2} - 2\overline{\frac{\partial p(x')}{\partial x'}\frac{\partial p(x)}{\partial x}}$$
$$= C_p\left(1 - A_p^o\left(\frac{r}{\eta}\right)^{\xi_{\mathscr{P}}-2}\right) \approx C_p = \text{const.}$$

Substituting (3) into (4), we find that in the inertial range

$$\xi_{2n,0} = (d-1)\left(\frac{(2n-1)S_{2n-2,2}}{S_{2n,0}} - 1\right) - (2n-1)\frac{\langle \delta p_x U^{2n-2}\rangle}{A_{2n,0} r^{\xi_{2n,0}-1}}. \tag{7}$$

The dependence of the functions $\xi_{n,m}$ upon the moment order $n+m$, which is intimately related to the geometric structure of turbulence, is one of the main unanswered questions. In particular, the asymptotics $n+m\to\infty$ is of special interest since it is related to the most violent turbulent fluctuations, responsible for the tails of probability density. Towards the elucidation of these issues, we first prove the following theorems:

THEOREM 1. *If in the limit of infinite Reynolds number* ($\eta\to 0$) *the dissipative ('Kolmogorov') length scales of the structure functions* $S_{2n,0}$, $S_{2n-2,2}$, $S_{4n-4,0}$ *and* S_{p2} *are of the same order* ($\eta_p \approx \eta_{2n,0} \approx \eta_{2n-2,2} \approx \eta_{4n-4,0} \approx \eta$), *then the linear relation* $\xi_n = \alpha n + \kappa$ *is possible only with* $\alpha \geqslant (\xi_{\mathscr{P}} - \kappa)/4$.

Proof. It follows from (7) that

$$\left|\xi_{2n,0} - 2\left((2n-1)\frac{A_{2n-2,2}r^{\xi_{2n-2,2}}}{A_{2n,0}r^{\xi_{2n|,0}}} - 1\right)\right| = (2n-1)\left|\frac{\langle \delta p_x U^{2n-2}\rangle}{A_{2n,0} r^{\xi_{2n,0}-1}}\right|. \tag{8}$$

It is important that all amplitudes $A_{m,n}$ on the left-hand side of (8) are independent of the Reynolds number or of η. By the Schwartz inequality, we have

$$\left|\xi_{2n,0} - 2\left((2n-1)\frac{A_{2n-2,2}r^{\xi_{2n-2,2}}}{A_{2n,0}r^{\xi_{2n|,0}}} - 1\right)\right| \leqslant (2n-1)\frac{\sqrt{C_p A_{4n-4,0}}}{A_{2n,0}} r^{\Gamma}. \tag{9}$$

where

$$\Gamma = \frac{\xi_{4n-4,0}}{2} - \xi_{2n,0} + 1. \tag{10}$$

When $\eta \to 0$ ($C_p = O(\eta_p^{\xi_{\mathscr{P}}-2})$), the inequality (9) is trivially satisfied for all $r = O(1)$. However, we are interested in behaviour of structure functions in the entire inertial range $L > r = \mathscr{C}\eta \to 0$ where (9) is valid. Since in that region the amplitude C_p is η-dependent, the validity of the inequality (9) is not trivial.

First, we consider $\xi_{2n-2,2} - \xi_{2n} = \beta_{2n} \geqslant 0$. In this case, for $r \approx \mathscr{C}\eta \to 0$, the left-hand side of (9) tends to the η-independent non-zero constant. Thus, the relation (9) holds for these values of r only if the non-zero pressure-gradient term $(O(\sqrt{C_p}\eta^{\Gamma}))$ on the right-hand side of (9) is such that

$$\Gamma + \frac{\xi_{\mathscr{P}} - 2}{2} \leqslant 0.$$

Substituting $\xi_n = \alpha n + \kappa$ into the above relations† gives

$$\alpha \geqslant \frac{\xi_{\mathscr{P}} - \kappa}{4}. \tag{11}$$

Now, we consider $\beta_{2n} < 0$. In this case, the left-hand side of (9) tends to infinity and the inequality is valid if and only if

$$-|\beta_{2n}| \geqslant \frac{\xi_{\mathscr{P}} - 2}{2} + \frac{\xi_{4n-4,0}}{2} - \xi_{2n,0} + 1.$$

Again, substituting $\xi_n = \alpha n + \kappa$ gives

$$\alpha \geqslant \frac{\xi_{\mathscr{P}} - \kappa + 2|\beta_{2n}|}{4} \geqslant (\xi_{\mathscr{P}} - \kappa)/4.$$

Thus, the inequality (11) is valid disregarding the relation between the exponents $\xi_{2n-2,2}$ and $\xi_{2n,0}$. □

By resorting to a single dissipation scale $\eta \propto Re^{-3/4}$, the Kolmogorov theory discarded the possible fluctuations‡ of η. The above result shows that if we accept this single-scale concept, Kolmogorov's scaling relation $\xi_{n,0} = n/3$ does not contradict the Navier–Stokes equations, provided $\xi_{\mathscr{P}} = 4/3$ and $\kappa = 0$. This will be particularly important in the case of two-dimensional turbulence in the inverse cascade range, where the smallest scale is fixed by the pumping force scale η_f and $\eta_n \approx \eta_p \approx \eta_f$. Since in the two-dimensional case $S_{3,0} \propto r$, we have $\alpha = 1/3$ and $\xi_p = 4/3$ in accordance with the quasi-normal calculations of Monin & Yaglom (1971).

THEOREM 2. *In the limit $n \to \infty$, the dissipation scales $\eta_{2n,0} \leqslant LRe^{-1/2}$.*

Proof. Since $A_{2n,0} = O(1)$, according to (3),

$$\eta_{2n,0} \approx \overline{(\partial_x u(0))^{2n}}^{\frac{1}{\xi_{2n,0}-2n}}.$$

The dissipation rate $\epsilon \approx \nu\overline{(\partial_x u(0))^2} = O(1)$, and so by the Schwartz inequality,

$$\eta_{2n,0} \leqslant \overline{(\partial_x u(0))^2}^{\frac{n}{\xi_{2n,0}-2n}} \approx \nu^{-\frac{n}{\xi_{2n,0}-2n}} \approx Re^{\frac{n}{\xi_{2n,0}-2n}} \equiv Re^{s_{2n}}.$$

Since $\xi_n > 0$ and $\xi_{2n,0} - 2n < 0$, when $n \to \infty$, the allowed minimum of the difference $\xi_n - n \to -n$ and as $Re \to \infty$, the dissipation scales $\eta_{2n,0}$, satisfy the inequality

$$\eta_{2n,0} \leqslant Re^{-1/2}.$$

□

THEOREM 3. *In accordance with the theory of multifractal random processes (Frisch 1995), as $n \to \infty$, $\eta_{2n,0} \to Re^{s_n}$ with $-\frac{3}{4} \geqslant s_n \geqslant -1$ independent of the moment order n. If $\xi_n/n \to 0$, then $s_n \to -1$.*

† G. Eyink drew my attention to the importance of the subleading contribution of κ to the linear asymptotics $\xi_n = \alpha n$.

‡ The notion of a continuum of dissipation scales has been considered in other contexts by Paladin & Vulpiani (1987) and by Sreenivasan & Meneveau (1988).

Proof. According to the theory of multifractality (Frisch 1995) $\overline{(\partial_x u(x))^n} \propto Re^{\zeta_n+n/2}$ where $\zeta_n = p(n)-3n/2$. Here, the function $p(n)$ is a solution to the algebraic equation $\xi_p+p(n) = 2n$, where ξ_p is the exponent of the pth-order structure function introduced in (6). In the limit $n \to \infty$, this equation can be satisfied only when $p \to \infty$. If the ratio $\xi_p/p \to 0$ in the limit of large p, we will have $p(n) \to 2n$. As a result, $\zeta_n \to n/2$. Substituting this into the above expression for $\eta_{2n,0}$ gives

$$\eta_{n,0} \propto Re^{\frac{n}{\xi_{n,0}-n}} \to Re^{-1}.$$

Another case of substantial interest is the limit $\xi_p/p \to q \leqslant 1/3$. Then, $\eta_{n,0} \propto Re^{\frac{p(n)-n}{\xi_p-n}} = Re^{\frac{1-q}{q^2-1}}$, which for $q = 1/3$ gives the expected Kolmogorov dissipation scale $\eta_n \propto Re^{-3/4}$. □

Similar considerations lead to the conclusion that as $n \to \infty$, the dissipation scale $\eta_{2n-2,2} = O(\eta_{2n,0})$. (They are not necessarily all equal.) The above results show that in the limit $n \to \infty$, the dissipation scales of the even-order structure functions $S_{2n,0}$, $S_{4n-4,0}$ and $S_{2n-2,2}$ are of the same order in Re. However, in this limit, the exponent of the pressure structure functions $\xi_{\mathscr{P}} \neq \xi_{2n,0}$ and, to complete the argument, we have to assess the modifications to the inequality (11) when $\xi_{\mathscr{P}} \approx \xi_{4,0} \neq \xi_{2n,0}$. The theory of multifractality (see Frisch 1995) gives for $\eta_{4,0} = O(Re^{-0.76})$ (close to Kolmogorov's value 3/4), so that, as $Re \to \infty$, $\eta_p \approx \eta_{2n,0}^{0.75}$. A similar conclusion ($\eta_p = O(Re^{-3/4})$) has been reached in Gotoh & Nakano (2003) as a result of their numerical simulations. Substituting this into (9) and using $\xi_{\mathscr{P}} \approx \xi_{4,0} \approx 4/3$, we obtain

$$\alpha \geqslant 0.375 - \frac{\kappa}{4}. \tag{12}$$

This result was obtained for the asymptotics $\eta_{n,0} \propto 1/Re$. In the second limiting case $\eta_p \approx \eta_n \approx Re^{-3/4}$ and $\alpha \geqslant (\xi_{\mathscr{P}} - \kappa)/4$, derived above.

The following remarks pertain to (12). According to the theory of multifractal processes, $\kappa = 3 - D$, where D is the fractal dimension of the most singular structure of turbulence. Thus, the inequality (12) is a dynamic constraint relating scaling exponents ξ_n to the dimension D.

To conclude this section, let us consider the large-n asymptotics (see below):

$$\xi_n = \xi_\infty - \frac{b}{n} \to \xi_\infty = \text{const}$$

where the saturation exponent ξ_∞ and b are the Reynolds-number-independent constants. Repeating the above simple calculation gives the inequality for the saturation exponent ξ_∞:

$$-\frac{\xi_\infty}{2} + \frac{3\xi_{\mathscr{P}}}{8} + \frac{1}{4} \leqslant 0$$

which for $\xi_{\mathscr{P}} = 4/3$ gives $\xi_\infty \geqslant 3/2$.

3. A model for the pressure contributions

The pressure–velocity correlation functions are very hard to determine experimentally. Instead, as suggested by Kurien & Sreenivasan (2001), one can measure the structure functions $S_{n,m}$ directly and deduce the pressure contributions from an exact equation (4) as

$$G_{2n} = -1 + \frac{\dfrac{(d-1)(2n-1)}{r} S_{2n-2,2}}{\dfrac{\partial S_{2n,0}}{\partial r} + \dfrac{d-1}{r} S_{2n,0}} = \frac{(2n-1)\langle \delta p_x (\delta u)^{2n-2} \rangle}{\dfrac{\partial S_{2n,0}}{\partial r} + \dfrac{d-1}{r} S_{2n,0}}. \tag{13}$$

Recently, in an important paper Gotoh & Nakano (2003) reported the results of a detailed numerical investigation of the contributions to equation (4). They demonstrated that the dissipation contributions to (4) are indeed negligible and the conditional expectation value of the pressure gradient difference $\langle \delta p_x | U \rangle = O(U^2/r)$. With remarkable accuracy Gotoh & Nakano showed that in the inertial range the ratios $G_{2n}(r) \approx \text{const}$.

Gotoh & Nakano (2003) went further. Assuming a vortex tube as a dominant structure and using Bernoulli's equation, they evaluated the conditional expectation of the pressure gradient as

$$-\langle \delta p_x | U \rangle \approx a \partial_r U^2, \tag{14}$$

where the constant factor $a > 0$ dominates the pressure contribution to equation (4) for $2n > 2$. When $2n = 2$, the pressure term (5) is equal to zero.

Then, assuming $\partial_r U^2 = O(U^2/r)$, Gotoh & Nakano (2003) obtained

$$-\langle (\delta p_x) U^{2n-2} \rangle = \frac{\hat{a}}{r} \left(S_{2n,0} - \frac{S_{3,0}}{S_{2,0}} S_{2n-1} - S_{2,0} S_{2n-2,2} \right). \tag{15}$$

The three contributions to the right-hand side of this relation are proportional, respectively, to $r^{\xi_{2n,0}}$, $r^{1-\xi_{2n,0}+\xi_{2n-1,0}}$ and $r^{\xi_2+\xi_{2n-2,0}}$. Due to intermittency $\xi_{2n,0} < 1 - \xi_{2,0} + \xi_{2n-1}$ and $\xi_{2n,0} < \xi_{2,0} + \xi_{2n-2,0}$, and, in the limit $r \to 0$, the first term in (15) dominates all the relations with $2n > 2$. In this way Gotoh & Nakano (2003) obtained a homogeneous equation for the structure functions. The magnitude of the coefficient calculated by them was $\hat{a} \approx 0.29$. Thus, the problem of anomalous scaling of longitudinal structure functions was reduced to the problem of pressure–velocity correlation functions. In principle, the coefficient a can be a function of r. Setting $a \propto r^{\gamma}$, they concluded that if $\gamma > 0$, the pressure contribution disappears in the limit $r \to 0$. If $\gamma = 0$, then $\xi_{2n,0} = \xi_{2n-2,2}$. This is the case we will analyse in what follows. In the limit $n \to \infty$, model (15) combined with relation (4) gave $\xi_{2n,0} \approx 0.58n$ which is in strong contradiction with the data on three-dimensional intermittent turbulence.

Thus, while the constancy of G_{2n} in the inertial range is consistent with the Gotoh–Nakano model, the inconsistency just mentioned suggests that it needs some changes. In what follows, we abandon the assumption leading to (15) and investigate equation (4) with a somewhat modified model for the pressure contribution:

$$-\langle \delta p_x | U, V \rangle \approx a \partial_r U^2 - b \frac{V^2}{r}. \tag{16}$$

A new element in the model (16), which for $d = 3$ is to be used for $2n > 2$ (see below), is that we consider the expectation value of the pressure gradient difference conditioned by the fixed values of both U and V.† The model (16) generates corrections to the coefficients in front of various contributions to the exact equation (4) for the generating function.

† J. Dovoudi drew my attention to this aspect, and found a similar feature in the case of two-dimensional turbulence.

With (16) equation (4) becomes

$$\frac{\partial S_{2n,0}}{\partial r}+\frac{2}{r}S_{2n,0}=\frac{2(2n-1)}{r}S_{2n-2,2}+(2n-1)\left\langle\left[a\frac{\partial(\delta u)^2}{\partial r}-b\frac{(\delta v)^2}{r}\right](\delta u)^{2n-2}\right\rangle, \quad (17)$$

or, equivalently,

$$\frac{\partial S_{2n,0}}{\partial r}+\frac{2}{r}S_{2n,0}=\frac{2(2n-1)}{r}S_{2n-2,2}+\frac{(2n-1)}{n}a\frac{\partial S_{2n,0}}{\partial r}-b(2n-1)\frac{S_{2n-2,2}}{r}. \quad (18)$$

Substituting (3) with $\xi_{2n-2,0}=\xi_{2n,0}$ into (18) gives

$$\xi_{2n,0}=\frac{(2n-1)(2-b)(A_{2n-2,2}/A_{2n,0})-2}{(1-2a)n+a}n. \quad (19)$$

Under these assumptions, the problem of the scaling exponents of the structure functions is reduced to information about the amplitude ratio $R_{2n,0}^{2n-2,2}\equiv A_{2n-2,2}/A_{2n,0}$. In a purely Gaussian case $(2n-1)R_{2n,0}^{2n-2,2}=S_{0,2}/S_{2,0}\approx 4/3$. (Actually, $(2n-2)R_{2n,0}^{2n-2,2}=(\xi_{2,0}+2)/2\approx 1.35$.)† It was argued in Yakhot (1998) that in the inertial range $(2n-1)R_{2n,0}^{2n-2,2}\approx 4/3$ is a universal constant and deviations from the Gaussian statistics of velocity differences originate from anomalous scaling exponents. Strictly speaking, expression (19) was derived for the even-order moments only. Assuming that it is also valid for the odd-order moments, and recalling that $\xi_3=1$, one obtains the relation between the coefficients a and b to be $a-b=1/4$. Then, with $a\approx 0.473$ and $b\approx 0.22$ we have

$$\xi_{2n,0}\approx\frac{0.185}{0.473+0.055n}2n \quad (20)$$

derived in Yakhot (2001). This expression is in an excellent agreement with experimental data (see Kurien & Sreenivasan 2001 for a comparison) and with predictions of Yakhot (2001) obtained for transverse structure functions $S_{0,2n}$. Note, however, that the pressure contributions do not enter a purely kinematic relation for the second-order moment, and thus model (16) must include some additional terms forcing $\langle\delta p_x\rangle=0$. This can be easily done in the manner in which the model (15) is constructed. It is clear, however, that, as in relation (15), the additional contributions disappear in the inertial range of the higher-order moments. Thus, strictly speaking, model (16) is to be used exclusively for the moments with $2n\geqslant 4$.

It follows from (4) and (18) that the ratios G_{2n} investigated in Kurien & Sreenivasan (2001) and in Gotoh & Nakano (2003) are

$$G_{2n}=\frac{(2n-1)\langle\delta p_x(\delta u)^{2n-2}\rangle}{\dfrac{\partial S_{2n,0}}{\partial r}+\dfrac{2}{r}S_{2n,0}}\approx\frac{0.473\dfrac{2n-1}{n}\xi_{2n,0}-0.29}{\xi_{2n,0}+2} \quad (21)$$

(noting that $(2n-1)R_{2n,0}^{2n-2,2}b\approx 1.33\times 0.22\approx 0.29$), giving $G_4\approx 0.19$, $G_6\approx 0.29$ and $G_8\approx 0.37$, in good agreement with numerical simulations of Gotoh & Nakano (2003) and experimental data in the atmospheric boundary layer of Kurien & Sreenivasan (2001). The fact that, in the inertial range, the ratio $(2n-1)A_{2n-2,2}/A_{2n,0}\approx$ const

† Recently, T. Gotoh has kindly informed me that his numerical results give $(2n-1)R_{2n,0}^{2n-2,2}\approx 1.37$, which is close to $1.35\approx 4/3$.

has been reasonably well confirmed experimentally (Sreenivasan 1997, personal communication; Gotoh & Nakano 2003).

4. Conclusions

The main contributions of the work are the following: (i) We have shown that if, in the limit $Re \to \infty$, the dissipation scales of various-order structure functions and of the second-order moment of the pressure difference are of the same order, then the linear large-n asymptotics $\xi_{n,0} = \alpha n + \kappa$ is possible only if the proportionality coefficient $\alpha \geqslant (\xi_{\mathscr{P}} - \kappa)/4$. Kolmogorov turbulence corresponds to the case $\kappa = 0$, corresponding to the fractal dimension $D = 3$. (ii) In the limit $Re \to \infty$, the dissipation scales of velocity structure functions satisfy $\eta_{2n,0} \leqslant Re^{-1/2}$. (iii) In general, as $n \to \infty$, the linear asymptotics $\xi_{n,0} = \alpha n + \kappa$ is in accord with both the Navier–Stokes equations and theory of multifractal processes only if $\frac{3}{8} - \frac{\kappa}{4} \leqslant \alpha$. Since $\kappa = 3 - D$, this inequality is a dynamic constraint relating the scaling exponents to the fractal dimension of the most singular feature of turbulence. (iv) The results of numerical experiments of Gotoh & Nakano (2003) on the pressure contributions to the equations for the longitudinal structure functions $S_{2n,0}$ are consistent with asymptotic saturation of scaling exponents $\xi_{2n} \to$ const in the limit $n \to \infty$. (v) The concept of universality assumes the independence of the inertial-range scaling exponents from the flow details. The amplitudes $A_{n,m}$ are not supposed to be universal. It follows from expressions (4), (18) and (19) that the magnitudes of the scaling exponents explicitly depend upon the ratios of the amplitudes $R_{2n,0}^{2n-2,2} = A_{2n-2,2}/A_{2n,0}$ which, if universality of turbulence does exist, must be universal. This statement agrees with the exact kinematic relation $A_{0,2}/A_{2,0} = (\xi_{2,0} + d - 1)/(d - 1)$.

One of the results of this work deserves a special mention. The fact that there exists an entire spectrum of the 'dissipation scales' $\eta_{n,0} \approx Re^{s_n}$ with $-1/2 \geqslant s_n \geqslant -1$ means that the most violent velocity fluctuations are characterized by the length scale $\eta_{n,0} \ll LRe^{-3/4}$. This places a severe constraint on the resolution requirements of direct numerical simulations of turbulence. It clear that all existing simulations based on the mesh-size $\Delta \approx \eta \approx LRe^{-3/4}$ cannot accurately predict the properties of the violent structures of turbulence. This important problem will be the subject of future communications.

I am grateful to T. Gotoh for helpful discussions and numerical verification of some of the assumptions used in this paper. My thanks are due to A. Polyakov, K. R. Sreenivasan, Y. Sinai, P. Constantin, G. Falkovich, G. Eyink, S. Y. Chen and C. Meneveau for important suggestions, encouragement and interest in this work.

REFERENCES

Frisch, U. 1995 *Turbulence*. Cambridge University Press.

Gotoh, T. & Nakano, T. 2003 Role of pressure in turbulence. *J. Statist. Phys.* (in press).

Hill, R. 2001 Structure functions of all orders. *J. Fluid Mech.* **434**, 379–388.

Kolmogorov, A. N. 1941 The local structure of turbulence in incompressible viscous fluids for very large Reynolds number. *Dokl. Acad. Nauk SSSR* **30**, 9–13 (Reprinted in *Proc. R. Soc. Lond.* A **434**, 9–13.

Kurien, S. & Sreenivasan, K. R. 2001 Experimental results on a new dynamical equations of turbulence. *Phys. Rev.* E **64**, 056302.

Monin, A. S. & Yaglom, A. M. 1971 *Statistical Fluid Mechanics*, vol. 2. MIT Press.

Paladin, G. & Vulpiani, A. 1987 Anomalous scaling laws in multifractal objects. *Phys. Rep.* **156**, 147–157.

Sreenivasan, K. R. & Meneveau, C. 1988 Singularities of the equations of fluid motion. *Phys. Rev.* A **38**, 6287–6295.

Yakhot, V. 1998 Probability density and scaling exponents of the moments of longitudinal velocity difference in strong turbulence. *Phys. Rev.* E **57**, 1737

Yakhot, V. 2001 Mean-field approximation and a small parameter in turbulence theory. *Phys. Rev.* E **63**, 026307.